# Algebra and geometry of link homology

## Plan:

**Lecture 1**: Definition of Khovanov-Rozansky (HOMFLY-PT) homology, examples
(mostly commutative algebra)

**Lecture 2**: Braid varieties
(geometric model for lecture 1)

**Lecture 3**: y-ification, tautological (homological operations + deformations) classes, more examples.

**Lecture 4**: Algebraic links, singular curves and affine Springer theory
(another geometric model, related to J. Kamnitzer's course)

**Lecture 5**: $\mathrm{Hilb}^n(\mathbb{C}^2)$ and link homology.

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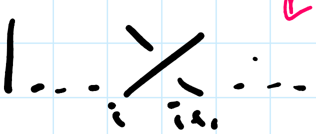
## What is a link invariant?

Braid group: gens $\sigma_1 \ldots \sigma_{n-1}$

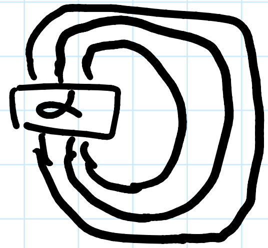



rels $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

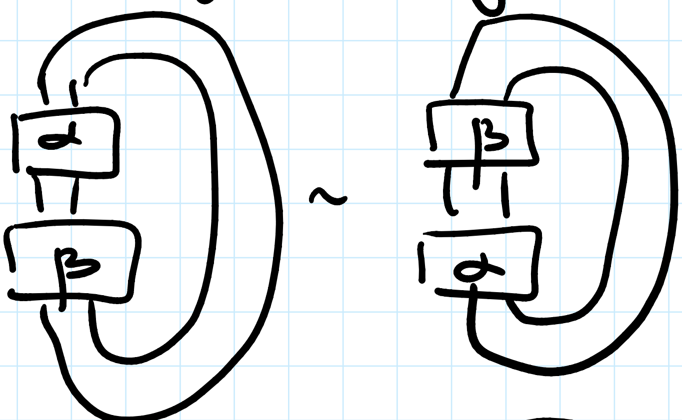
$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i-j|>1$

<u>Thm</u> (Alexander) Any link is a closure of a ... braid

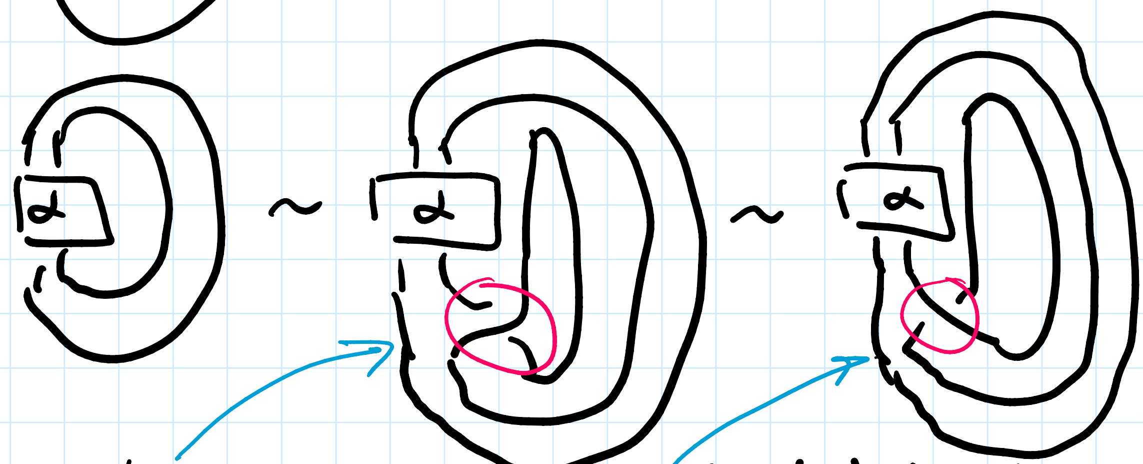
**Thm** (Alexander) Any link is a closure of some braid.




**Thm** (Markov) Two braids close to the same link if and only if they are related by the moves:




(conjugation)




positive (resp. negative) stabilization
[Note: changes number of strands!]

**Conclusion**: to build a top. link invariant, it is sufficient to:

- Assign something to crossings $\sigma_i^{\pm}$
- Verify braid relations $\Rightarrow$ invariant of a braid

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- Describe an operation for closing a braid ("trace"-like")
- Verify invariance under conjugation + stabilization.

**Rmk** Sometimes it is useful to consider weaker invariants (say, only conjugation and positive stabilization).

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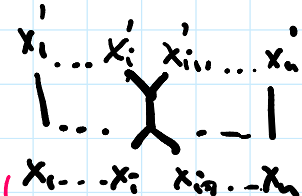
$R = \mathbb{C}[x_1 \ldots x_n] \circlearrowleft S_n$

$B_i = R \underset{R^{s_i}}{\otimes} R$ $\leftarrow$ R-R bimodule

$s_i = (i\ \ i{+}1)$




**Exercise**: $B_i \underset{R}{\otimes} B_i = B_i \oplus B_i$

$x_i + x_{i+1} = x_i' + x_{i+1}'$

$x_i x_{i+1} = x_i' x_{i+1}'$

$x_j = x_j'$ $j \neq i, i{+}1$

**Exercise**: There are interesting maps of bimodules $B_i \to R$ and $R \to B_i$

$T_i = [B_i \to R]$ $\quad T_i^{-1} = [R \to B_i]$

Complexes of R-R bimodules.

**Thm** (Rouquier) $T_i, T_i^{-1}$ satisfy braid rels. up to homotopy

$T_i \otimes T_i^{-1} \simeq R \qquad T_i \otimes T_{i+1} \otimes T_i \simeq T_{i+1} \otimes T_i \otimes T_{i+1}$

up to homotopy

$$T_i \otimes T_i^{-1} \simeq R \qquad T_i \otimes T_{i+1} \otimes T_i \simeq T_{i+1} \otimes T_i \otimes T_{i+1}$$

$$T_i \otimes T_j \simeq T_j \otimes T_i \qquad |i-j|>1$$

**Cor** $\beta$ = braid on $n$ strands

$\rightsquigarrow T_\beta$ = complex of $R$-$R$ bimodules well defined up to homotopy.

**Rmk:** $B_i$ and their tensor products generate the category of **Soergel bimodules**

$\Rightarrow T_\beta$ is a complex of Soergel bimodules.

**Rmk:** Can be defined for any Coxeter group & corresponding braid group.

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Braid closure $\Longrightarrow$ Hochschild homology (termwise)

$$\beta \rightsquigarrow T_\beta \longrightarrow HH(T_\beta) \rightsquigarrow \text{take homology}$$

complex of $R$-bimodules

complex of $R$-modules

$$H(HH(T_\beta)) = HHH(\beta).$$

**Thm** (Khovanov, Rozansky) $HHH(\beta)$ is a link invt, that is, invt under conjugation and stabilization.

and stabilization.

<u>Triply graded</u>: $Q$ = internal grading on $B$: $\deg(x_i)=2$

$T$ = homological grading

$A$ = Hochschild grading

<u>Rmk</u> $HH^0(X) = \mathrm{Hom}(R,X)$ ⇝ we will often restrict to this part of HHH. This is invariant under conjugation & positive stabilization but <u>not</u> invariant under negative stabilization.

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<u>Examples</u>:  $T = [B \to R]$

$$T^2 = [B \to R] \otimes [B \to R] =$$

$$= [B^2 \to B \oplus B \to R] =$$

$$= [B \oplus B \to B \oplus B \to R] =$$

$$= [B \to B \to R]$$

Apply $\mathrm{Hom}(R,-)$: $\mathrm{Hom}(R,B)=R$

$$\mathrm{Hom}(R,T^2) = [R \xrightarrow{0} R \xrightarrow{x_1-x_2} R]$$

⇝ compute homology.

More 2-strand examples in exercises.

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**Problem:** $T_\beta$ grows exponentially in the number of crossings, how to compute?

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**Thm** (Elias, Hogancamp, Mellit)

HHH (all positive torus links) is supported in <u>even</u> homological degrees + explicit recursion computing Poincaré poly.

**Ex**: $HHH(T(n,n+1)) = q,t$-Catalan

Conjectured by G., Negut, Oblomkov, Rasmussen, Shende..

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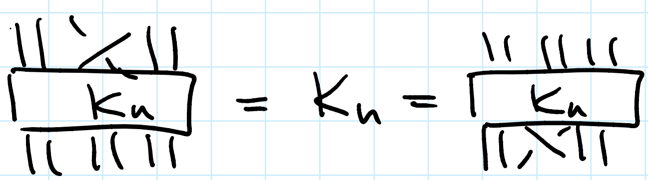
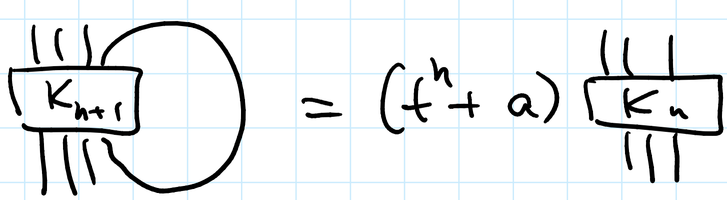
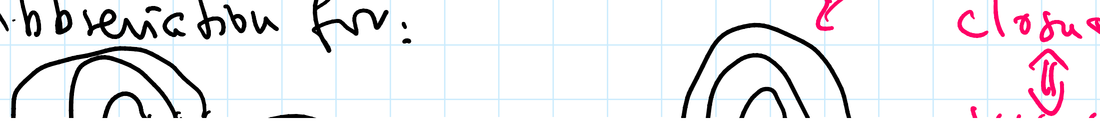
**Technology:** **Thm** (Hogancamp) $\exists$ complex of Soergel bimodules $K_n$ such that: $\boxed{\text{$K_n$ = compact categorical Jones–Wenzl projector}}$

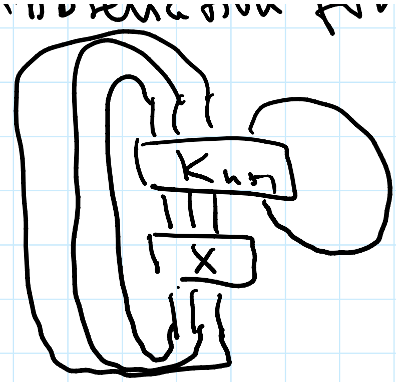
-  (each crossings)

- 

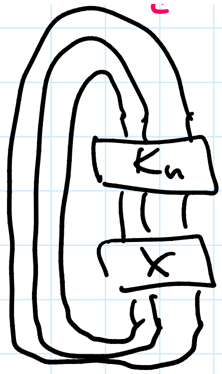
Abbreviation for: 

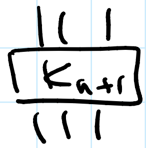
braid closure

...relation for: closure ⟺ HHH

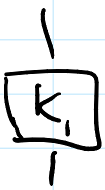


$= (t^n + a)$



•  $= t^{-n}\Big($  $\longrightarrow q$  $\Big)$

There is a chain map here such that its cone is homotopy equivalent to the complex on the left.

•  $=$ |

a, q, t = grading shifts

$q = Q^2$
$t = T^2 Q^{-2}$
$a = A\,Q^{-2}$

A, Q, T = standard HOMFLY gradings.

even homological (T) degree.

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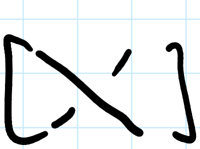
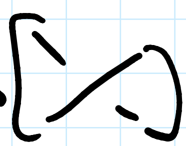
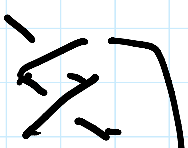
**Ex:** $K_2 = [R \to B \to B \to R]$

$=$  $\longrightarrow$  $=$

$= [\,||\, \longrightarrow$  $]$

Exercises/Q&A sessions: How to use this thm

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